The moduli space of two U(1) instantons
on noncommutative R^4 and $R^3 \times S^1$.

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We employ the ADHM method to derive the moduli space of two instantons in $U(1)$ gauge theory on a noncommutative space. We show by an explicit hyperKähler quotient construction that the relative metric of the moduli space of two instantons on R^4 is the Eguchi-Hanson metric and find a unique threshold bound state. For two instantons on $R^3 \times S^1$, otherwise known as calorons, we give the asymptotic metric and conjecture a completion. We further discuss the relationship of caloron moduli spaces of A, D and E groups to the Coulomb branches of three dimensional gauge theory. In particular, we show that the Coulomb branch of $SU(2)$ gauge group with a single massive adjoint hypermultiplet coincides with the above two caloron moduli space.

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1   Introduction and Conclusion

Recently there has been a great deal of interest in field theories on non-commutative spaces and their classical soliton solutions [1, 2]. Examples include instantons, monopoles, vortices, CP(n) solitons, as well as novel solitons which only exist in non-commutative theories [3, 4, 5, 6, 8, 7]. While most recent studies have focused on the characteristics of the solitonic field configurations themselves, there have also been several works investigating the low energy dynamics and, in particular, the moduli spaces of these solitons. A key point is that the moduli spaces are ordinary Riemann spaces; they feel the effect of the non-commutivity of the underlying spacetime only through changes to their metric.

For example, the moduli space of a single instanton in a $U(n)$ gauge theory on non-commutative R^4 or $R^3 \times S^1$ has been identified by one of us (K.L.), together with P.Yi in Ref. [9]. For a single instanton on $R^3 \times S^1$ - otherwise known as a caloron - simplifications occur when a Wilson line on the circle breaks the gauge group to the maximal torus. In this case, the asymptotic form of the moduli space may be derived by considering the dynamics of constituent monopoles from which the caloron is constructed [10]. The resulting metric includes several parameters corresponding to the radius, R, of the S^1 factor, the value of the Wilson line, as well as the non-commutivity parameter itself. For all values of these parameters, the metric is hyperKähler and complete and, in analogy with monopoles, is believed to be exact. Moreover, the metric has a smooth limit in the decompactification limit $R \to \infty$. After separating the center-of-mass motion, the remaining $4(N-1)$-dimensional metric is known as the Calabi metric: as expected of the non-commutative $U(N)$ instanton moduli space, it has a tri-holomorphic $SU(N)$ isometry, together with a non tri-holomorphic $U(1)$ isometry. For the case of $N = 2$, the Calabi metric coincides with the Eguchi-Hanson metric, which arises as the blow-up of R^4/Z_2.

In this paper, we further explore the moduli spaces of instantons. We firstly consider two instantons in a non-commutative $U(1)$ gauge theory. In the ordinary, commutative limit, the instantons become singular and their moduli space is simply the space of two unordered points on R^4. After separating the center-of-mass motion, this leaves the relative moduli space R^4/Z_2. The effect of the non-commutative spacetime is to resolve this singularity and the metric on the relative moduli

space of two $U(1)$ instantons is again expected to be Eguchi-Hanson[1]. In the first part of this paper, we confirm this conclusion by performing an explicit hyperKähler quotient construction of the metric using the ADHM method. Note that in this case, the quotient is performed with a non-abelian group. We also find that there exists a unique threshold bound state of these two U(1) instantons, in context of $N = 4$ supersymmetric five dimensional theory. When six dimensional (1,1) supersymmetric theory is compactified on a circle, a single instanton can be regarded as a mode of winding number one. The threshold state can be regarded as a mode of winding number two.

For two $U(1)$ non-commutative instantons on $R^3 \times S^1$, we derive the asymptotic form of the metric and observe that it coincides with the $U(1)$ hyperKähler quotient of the relative eight dimensional moduli space for one massless and two massive monopoles in $SO(5)$ gauge theory, whose metric was found explicitly sometime ago by one of us (K.L) and C. Lu [12]. In addition, both metrics become flat as the deformation parameter vanishes. We conjecture that this correspondence extends to the full metric.

We further describe how these metrics appear as the quantum corrected Coulomb branches of certain $\mathcal{N} = 4$ three dimensional gauge theories. In particular, we show that the Coulomb branch of $SU(N)$ with a single massive adjoint hypermultiplet[2] is given by the moduli space of N $U(1)$ instantons on $R^3 \times S^1$. The gauge coupling constant determines the radius R of the S^1. Moreover, we describe the gauge theories yielding the moduli space of calorons for all simply-laced gauge groups and explain how these results tie in with known ideas of mirror symmetry in three dimensional gauge theories.

The plan of this paper is as follows. In Sec.2, we briefly review the ADHM formalism [14]. In Sec.3, the moduli space metric of two $U(1)$ instantons on non-commutative R^4 is obtained explicitly by a hyperKähler quotient construction. In section 4 we discuss the moduli space of two non-commutative calorons. In section 5, we discuss the vacuum moduli spaces of three dimensional gauge theories and their relationship to the moduli spaces of various classical solitons.

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[1]This point was also made recently in [11].

[2]After submitting this paper we were informed that this observation was previously made by Kapustin and Sethi [13].

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2   The ADHM Formalism

The ADHM approach to instanton physics arises naturally as the effective field theory describing k D0 branes near n parallel D4 branes in type IIA string theory. The low energy dynamics of the D4 branes is described by a $U(n)$ gauge theory in 4+1 dimensions with 16 supercharges. The D0 branes appear as instantons within the D4 branes. The dynamics of D0 branes is given by quantum mechanics model with 8 supersymmetries which can be obtained from the dimensional reduction of $N = 1$ supersymmetric 6 dimensional gauge theory with gauge group $U(k)$, one adjoint hypermultiplet and n fundamental hypermultiplets. The D-terms describing the Higgs branch of the D0-brane theory coincide with the ADHM constraints and, after modding out by the $U(k)$ gauge symmetry, give a hyperKähler quotient description of the instanton moduli space. We denote the moduli space of k $U(n)$ instantons as $\mathcal{M}_{k,U(n)}$.

When a uniform external NS-NS $B_{\mu\nu}$ field is present, with non-vanishing components of both indices along the longitudinal directions of the D4 branes, the field theoretic limit is the five dimensional gauge theory with the same gauge group and supersymmetry, but with the spatial R^4 becoming non-commutative,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} , \tag{1}$$

where $\mu, \nu = 1, \cdots, 4$ and $B_{\mu\nu}^{-1} \sim \theta_{\mu\nu}$. When $B_{\mu\nu}$ has a self-dual part, the D0 branes (which appear as anti-self-dual instantons) cannot escape the D4-branes, a fact which is reflected in the disappearance of the singularities in the instanton moduli space. From the perspective of the D0-brane gauge theory, the self-dual part of $B_{\mu\nu}$ induces a FI parameter. This modifies the classical Higgs branch metric, blowing up its singularities. We consider the selfdual case where nonvanishing components of $\theta^{\mu\nu}$ are

$$\theta^{12} = \theta^{34} = -\frac{\zeta}{4} . \tag{2}$$

Defining,

$$y^0 = x^4 + ix^3, \ \ y^1 = -i(x^1 - ix^2) , \tag{3}$$

we have the non-vanishing commutation relations,

$$[y^0, \bar{y}^0] = [y^1, \bar{y}^1] = \frac{\zeta}{2} . \tag{4}$$

The instanton moduli space, whether deformed by FI term or not, is a hyperKähler space and the metric has three complex structures and corresponding Kahler forms. The ADHM construction provides the standard method to find the metric and Kahler forms, both of which arise from the kinetic terms for the matter hypermultiplets in the $U(k)$ theory of the D0 brane world-volume theory. To find the moduli space of k instantons in $U(n)$ gauge theory, we start from two k dimensional complex square matrices B_0, B_1 and two n by k complex matrices $I^\dagger$ and J. They form $4k^2 + 4kn$ dimensional flat hyperKähler space $\mathcal{M}_0$ with the metric

$$ds^2 = 2 \, \mathrm{tr}_k(dB_0 \otimes_s dB_0^\dagger + dB_1 \otimes_s dB_1^\dagger + dI \otimes dI^\dagger + dJ^\dagger \otimes dJ) \,, \tag{5}$$

where $\otimes_s$ is the symmetic direct product. In this space, there exist three constant complex structures $\mathcal{I}_s$ with $s = 1, 2, 3$ such that $\mathcal{I}_r \mathcal{I}_s = -\delta_{rs} 1_{4k^2 + 4kn} + \epsilon_{rst} \mathcal{I}_r$, and the corresponding Kähler forms,

$$w_3 = i \, \mathrm{tr}_k(dB_0 \wedge dB_0^\dagger + dB_1 \wedge dB_1^\dagger + dI \wedge dI^\dagger - dJ^\dagger \wedge dJ),$$
$$w_1 - iw_2 = 2i \, \mathrm{tr}_k(dB_0 \wedge dB_1 + dI \wedge dJ) \,. \tag{6}$$

(For the given metric $ds^2 = g_{ab} dx^a dx^b$ and the covariantly constant complex structure $I^a_{(s)b}$, one constructs the Kahler form $w_s = \frac{1}{2} I_{(s)ab} dx^a \wedge dx^b$.)

There is a gauge group $U(k)$ on this space, under which B_0, B_1 belong to the adjoint representation and $I, J^\dagger$ to the fundamental representation. There are also three natural moment maps

$$\mu_r = [B_0, B_0^\dagger] + [B_1, B_1^\dagger] + II^\dagger - J^\dagger J,$$
$$\mu_c = [B_0, B_1] + IJ \,, \tag{7}$$

which map from $\mathcal{M}_0$ to the coadjoint orbit of $U(k)$. The moduli space of instantons on commutative space is given by the hyperKähler quotient, $\mathcal{M}_1 = \{\mu_r^{-1}(0) \cap \mu_c^{-1}(0)\}/U(k)$. This moduli space of dimension $4kn$ is singular at some points where some of instantons shrink to zero size. The moduli space of instantons on non-commutative space is a deformation of this moduli space due to the parameter $\zeta > 0$ and is defined by the hyperKähler quotient [3, 15]

$$\mathcal{M}_{k,U(n)} = \{\mu_r^{-1}(\zeta 1_k) \cap \mu_c^{-1}(0)\}/U(k) \,. \tag{8}$$

This space is no longer singular, reflecting the fact that instantons cannot shrink to zero size. The metric and complex structures on the instanton moduli space are naturally induced from the

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original metric (5) and complex structures (6) by the hyperKähler quotient process. This is the structure we are interested in.

For a given set of matrices B_0, B_1, I, J, one can define a (n+2k) by 2k matrix

$$\Delta(x) = \begin{pmatrix} I^\dagger & J \\ B_0^\dagger - \bar{y}_0 & B_1 - y_1 \\ -B_1^\dagger + \bar{y}_1 & B_0 - y_0 \end{pmatrix} . \tag{9}$$

The equations $\mu_r = \zeta, \mu_c = 0$ are identical to the ADHM constraint

$$\Delta^\dagger \Delta = f^{-1}(x) \otimes 1_2 , \tag{10}$$

where f is a nonsingular k dimensional square matrix. For each solution to the ADHM constraint, one can find $n + 2k$ by n matrix v satisfying the equation

$$\Delta^\dagger v(x) = 0 , \tag{11}$$

together with the normalization condition

$$v^\dagger v = 1_n . \tag{12}$$

The selfdual gauge field configuration is then

$$A = v^\dagger dv . \tag{13}$$

The gauge field and field strength appear as operators defined on the Hilbert space specified by creation operators and annihilation operators made of spatial coordinate. Even though their moduli space is a hyperKähler space without any singularity in the sense of the Riemann geometry, the gauge field strength, which is an operator on the Hilbert space, is still singular and needs some sort of projection, or a modification of the underlying background space. We feel this issue is not settled and needs further study [16].

3 Two Instantons on Non-Commutative R^4

For a single instanton of the $U(N)$ gauge theory, the moduli space on R^4 is made of the center of mass R^4, together with the $4(N-1)$ dimensional Calabi metric which describes the scaling and

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gauge symmetry [9]. The next most simple example is that of two identical $U(1)$ instantons on non-commutative R^4. To find their moduli space metric, we start with finding the ADHM data satisfying the ADHM constraint,

$$\mu_r = \zeta 1_k , \quad \mu_c = 0 , \tag{14}$$

for two instantons in U(1) gauge theory. The general solution for this equation with two instantons is known in another context [17]. The step behind this is rather simple. First, one takes the trace of these equation to get constraints on I, J, which are $J \cdot I = 0$ and $I^\dagger I - J J^\dagger = 2\zeta$. One may use the $U(2)$ gauge symmetry to set $I^\dagger = (\alpha, 0)$ and $J = (0, \beta)$. These two equations show that the only consistent solution has $J = 0$. This now restores the $U(2)$ symmetry and it may be employed once more, this time to simultaneously triangularize B_0, B_1. Solving the equations with triangularized B_0, B_1 and I yields the ADHM data with satisfies (14), given by $J = 0$ and

$$B_0 = w_0 1_2 + \frac{z_0}{2} \begin{pmatrix} 1 & \sqrt{\frac{2b}{a}} \\ 0 & -1 \end{pmatrix} ,$$

$$B_1 = w_1 1_2 + \frac{z_1}{2} \begin{pmatrix} 1 & \sqrt{\frac{2b}{a}} \\ 0 & -1 \end{pmatrix} ,$$

$$I = \sqrt{\zeta} \begin{pmatrix} \sqrt{1-b} \\ \sqrt{1+b} \end{pmatrix} , \tag{15}$$

where we have chosen I to be real using the remaining gauge transformation, and where the dimensionless parameter is given by

$$a = \frac{|z_0|^2 + |z_1|^2}{2\zeta} \geq 0 ,$$

$$b = \frac{1}{a + \sqrt{1 + a^2}} \leq 1 . \tag{16}$$

The parameters w_0, w_1 represent the center of the mass coordinate of two instantons, and the parameters z_0, z_1 represent the relative position between two instantons. They are the coordinates of the eight dimensional moduli space for two instantons.

There are couple of comments to be made before we proceed to calculate the metric on the moduli space. First, in the limit of large separation, or large a, the matrices B_0 and B_1 become diagonal. Their eigenvalues $w_i \pm z_i/2$ are the positions of the two instantons. Second, we have not exhausted the gauge degrees of freedom by choosing the above solution. Under an SU(2) transformation

$$U = i \begin{pmatrix} -b & \sqrt{2ab} \\ \sqrt{2ab} & b \end{pmatrix} , \tag{17}$$

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the relative coordinates z_0, z_1 change their sign and the matrix I is invariant. Thus the exchange of the two instanton positions is a part of the gauge transformation, by which we want to mod out. This is expected as two instantons are identical solitons and so their relative moduli space should be R^4/Z_2 in large separation. Third, in the limit $a = 0$ where the two instantons overlap, the relative parts of B_0 and B_1 have non-vanishing upper right corner, and I has a nonzero lower component. Thus, under $U(2)$ transformation

$$U = \begin{pmatrix} e^{i\beta} & 0 \\ 0 & 1 \end{pmatrix} , \tag{18}$$

the relative data space becomes $(z_0/\sqrt{a}, z_1/\sqrt{a}) \sim e^{i\beta}(z_0/\sqrt{a}, z_1/\sqrt{a})$. This $U(1)$ gauge group makes the data space a two space, $S^3/U(1) = S^2$.

The tangent vectors for this ADHM data are a sum of the differentials with eight parameters and infinitesimal gauge transformation,

$$\delta B_0 = dB_0 - i[\alpha, B_0] ,$$
$$\delta B_1 = dB_1 - i[\alpha, B_1]$$
$$\delta I = dI - i\alpha I ,$$
$$\delta J^\dagger = dJ^\dagger - i\alpha J^\dagger , \tag{19}$$

where α is a hermitian two by two matrix and the differential d acts on all eight parameters. The tangent vectors satisfy the linearized ADHM constraints automatically. In addition, they are also required to satisfy the linearized Gauss law,

$$[\delta B_0, B_0^\dagger] - [B_0, \delta B_0^\dagger] + [\delta B_1, B_1^\dagger] - [B_1, \delta B_1^\dagger] + \delta I I^\dagger - I \delta I^\dagger - \delta J^\dagger J + J^\dagger \delta J = 0 . \tag{20}$$

These equations can be put together into

$$[\delta B_0, B_0^\dagger] + [\delta B_1, B_1^\dagger] + \delta I I^\dagger = 0 , \tag{21}$$

which fixes the matrix α uniquely,

$$\alpha = \frac{ib}{2\sqrt{1+a^2}} \begin{pmatrix} \frac{\bar\partial a - \partial a}{1-b} & -\frac{\sqrt{2}\bar\partial a}{\sqrt{ba}} \\ \frac{\sqrt{2}\partial a}{\sqrt{ba}} & \frac{\bar\partial a - \partial a}{1+b} \end{pmatrix} , \tag{22}$$

where $\partial a = (\partial a/\partial z_i) dz_i$ and $\bar\partial a = (\partial a/\partial\bar z_i) d\bar z_i$.

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Now it is straightforward to write the explicit expression for the tangent vector. It turns out $\delta I = 0$ in our case and so the metric of the moduli space for two instantons is,

$$ds^2 = 2\text{tr}\left(\delta B_0 \delta B_0^\dagger + \delta B_1 \delta B_1^\dagger\right) . \tag{23}$$

The metric naturally splits into to the center of mass part and the relative motion part:

$$ds^2 = ds_{\text{cm}}^2 + ds_{\text{rel}}^2 , \tag{24}$$

where

$$ds_{\text{cm}}^2 = 2dw_i d\bar{w}_i , \tag{25}$$

and

$$ds_{\text{rel}} = \frac{\sqrt{1+a^2}}{a}dz_i d\bar{z}_i - \frac{1}{a^2\sqrt{1+a^2}}\frac{1}{2\zeta}\bar{z}_i dz_i z_j d\bar{z}_j . \tag{26}$$

Under the change of variables to radial and angular coordinates r, θ, φ and ψ,

$$z_0 = r\cos\frac{\theta}{2}\exp\frac{i}{2}(\psi+\varphi) ,$$
$$z_1 = r\sin\frac{\theta}{2}\exp\frac{i}{2}(\psi-\varphi) , \tag{27}$$

the relative metric becomes

$$ds_{\text{rel}}^2 = \frac{1}{\sqrt{1+\frac{4\zeta^2}{r^4}}}(dr^2 + \frac{1}{4}r^2\sigma_z^2) + \frac{1}{4}\sqrt{1+\frac{4\zeta^2}{r^4}}r^2(\sigma_x^2 + \sigma_y^2) , \tag{28}$$

where σ_i are the standard left SU(2) invariant one forms

$$\sigma_x = -\sin\psi d\theta + \cos\psi\sin\theta d\varphi ,$$
$$\sigma_y = \cos\psi d\theta + \sin\psi\sin\theta d\varphi ,$$
$$\sigma_z = d\psi + \cos\theta d\varphi , \tag{29}$$

which satisfy $d\sigma_z = \sigma_x \wedge \sigma_y$ and cyclic permutations thereof. For the angular variables on S^3/Z_2, the ranges are given by,

$$0 \le \theta \le \pi, \ 0 \le \varphi \le 2\pi, \ 0 \le \psi \le 2\pi , \tag{30}$$

in contrast to the S^3 case for which $0 \le \psi \le 4\pi$.

Near the origin $r \approx 0$, the relative metric in new radial variable $\rho = r^2$ becomes

$$ds_{\text{rel}} \approx \frac{1}{8\zeta}(d\rho^2 + \rho^2 d\psi^2) + \frac{\zeta}{2}(d\theta^2 + \sin^2\theta d\varphi^2) , \tag{31}$$

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which is the metric for $R^2 \times S^2$. As the range of ψ is $[0, 2\pi]$, this metric is nonsingular. Thus the metric is smooth in the limit where two instantons are coming together.

As the relative space is hyperKähler, there exist three complex structures and corresponding Kahler forms. These three Kähler forms becomes

$$w_3 = \frac{1}{2} \left(\frac{a}{\sqrt{1+a^2}} r dr \wedge \sigma_z + \frac{\sqrt{1+a^2}}{2a} r^2 \sigma_x \wedge \sigma_y \right) ,$$
$$w_1 - iw_2 = \frac{r}{2}(dr + ir\frac{\sigma_z}{2}) \wedge (\sigma_x + i\sigma_y) . \tag{32}$$

The $SU(2)$ transformations which leave one-forms $\sigma_x, \sigma_y, \sigma_z$ invariant, leave the metric and the Kähler forms invariant and are thus a triholomorphic isometry. In addition the $U(1)$ rotation, $\psi \rightarrow \psi + \epsilon$ leaves the metric and the Kahler form w_3 invariant. However, it does not leave $w_1 - iw_2$ invariant and so its action is not a triholomorphic isometry. This $U(1)$ symmetry is the surviving symmetry of the SO(3) symmetry which rotates the three FI parameters $\vec{\zeta}$.

This symmetry consideration suggests that one could have guessed the metric of two U(1) instantons on non-commutative space should be Eguchi-Hanson [18]: the moduli space should be hyperKähler and the relative part should be four dimensional. Moreover, asymptotically the relative metric should be R^4/Z_2. The rotational symmetry O(4) of R^4 should be broken to $SO(3) \times U(1)$ due to the FI term $\vec{\zeta}$. In addition the $SO(3)$ symmetry should be triholomorphic and the $U(1)$ homomorphic. There is only one such hyperKähler space, which is of course the Eguchi-Hanson metric.

With the change of coordinates,

$$u^4 = r^4 + 4\zeta^2 , \tag{33}$$

we get the standard form of the Eguchi-Hanson metric

$$ds^2 = \frac{du^2}{1 - \frac{4\zeta^2}{u^4}} + \frac{u^2}{4} \left(\sigma_x^2 + \sigma_y^2 + (1 - \frac{4\zeta^2}{u^4})\sigma_z^2 \right) . \tag{34}$$

In the five dimensional $\mathcal{N} = 2$ Yang-Mills theory, the low-energy dynamics of instantons inherits $N = 8$ supercharges. The ground state of the supersymmetric quantum mechanics is then given by

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a normalizable, self-dual harmonic form on the moduli space [19]. On the Eguchi-Hanson metric, there exists a unique form with these properties,

$$
\begin{aligned}
\Omega &= d\left(\frac{\sigma_z}{u^2}\right) \\
&= \frac{1}{u^3}(-2du \wedge \sigma_z + u\sigma_x \wedge \sigma_y)
\end{aligned}
\tag{35}
$$

modulo a constant factor. Moreover, this form is invariant under $SO(3) \times U(1)$; it can be interpreted as a threshold bound state of two $U(1)$ instantons on noncommutative space. The existence of this bound state has been advocated before in Ref. [20]. The instantons of the 16 supersymmetric five dimensional Yang-Mills theory are identified with the Kaluza-Klein momentum modes of the six dimensional (2,0) supersymmetric theory compactified on a circle. Thus the threshold bound state of two $U(1)$ instantons corresponds to the state of two KK momenta. (In the similar vein, this threshold bound state can be also interpreted as the single particle state for a single $U(2)$ instanton on the noncommutative space, corresponding to the state of the KK mode of the $(2, 0)$ theory with $U(2)$ gauge group.)

4   Two Instantons on non-commutative $R^3 \times S^1$

In general, instantons, or calorons, on $R^3 \times S^1$ with nontrivial Wilson loop along the circle can be interpreted as a charge neutral composite of constituent BPS monopoles [10]. This construction for general simply-laced gauge group will be reviewed in the following section. While this interpretation in terms of monopoles is true for non-commutative $U(n)$ gauge theory with $n \geq 2$, there is a subtlety in the $U(1)$ gauge theory [9]: rather than a magnetic monopole, the caloron now appears as a magnetic dipole. This is most simply seen in the T-dual version of the D0-D4 system, in which calorons appear as D-strings wrapping the dual circle, ending on D3-branes. In the presence of a background NS-NS B-field, the BPS D-string is no longer perpendicular to the D3-branes, but is tilted [4]. The string therefore spirals around the circle and its two ends lie at different points on the D3-brane, where it appears as a dipole.

More specifically, let us compactify the x^4 direction with radius R; $0 \leq x^4 \leq 2\pi R$. The non-

commutative ADHM equations then reduce to the non-commutative Nahm equation,

$$\frac{dT^i}{dt} - i[T_4, Ti] = \frac{i}{2}\epsilon_{ijk}[T^j, T^k] + \delta_{i3}\frac{\zeta}{2} + \text{tr}_2(\tau_i aa^\dagger)\delta(t - v) \tag{36}$$

with t in the interval $[-1/(2R), 1/(2R)]$. For a single caloron, the function $T_i(t)$ is the position of the D string, and $T_3(t)$ increases linearly in t with slope $\zeta/2$. The end points where this D string meets D3 branes appear as the positive and negative monopoles, now with a finite separation of $\zeta/2R$ along the x^3 direction. Thus, a single caloron appears as a magnetic dipole. Now let us consider the dynamics of two dipoles: we denote the relative position between two positive charge as $\mathbf{r}$. This, of course, is identical to that between two negative charges. The relative positions between positive and negative charges are then $\mathbf{r} \pm (\zeta/2R)\hat{\mathbf{z}}$. While it is not obvious how to get the asymptotic form of the metric for large separation of monopoles in the gauge theory on non-commutative space, let us calculate the asymptotic metric in most naive way. The asymptotic form of the metric is then determined just by their charge. From the T-dualized D3-D1 brane picture, the identical charges have magnetic repulsion and Higgs attraction, while the opposite charges with D1 strings coming out from D3 branes oppositely, have magnetic attraction and Higgs repulsion.

Thus, the metric in large separation obtained by the standard technique [21] becomes

$$ds^2 = R\left(U(\mathbf{r})d\mathbf{r}^2 + U^{-1}(\mathbf{r})(d\psi + \mathbf{W}(\mathbf{r}) \cdot d\mathbf{r})^2\right) , \tag{37}$$

where

$$U(\mathbf{r}) = \frac{1}{R} - \frac{2}{|\mathbf{r}|} + \frac{1}{|\mathbf{r} + (\zeta/2R)\hat{\mathbf{z}}|} + \frac{1}{|\mathbf{r} - (\zeta/2R)\hat{\mathbf{z}}|} \tag{38}$$

up to overall constant.

A metric whose asymptotic form is identical to that of Eq.(37) has appeared previously in the discussion of the relative moduli space of one massless and two massive monopoles in SO(5) gauge theory, which is spontaneously broken to $U(1) \times SO(3)$ [12]. The net magnetic charge is purely abelian and the relative moduli space is eight dimensional. One can take the hyperKähler quotient of this eight dimensional metric with the unbroken $U(1)$ triholomorphic gauge symmetry of unbroken $SO(3)$ to get a four dimensional metric. Physically one may think of this as holding the position of the massless monopole fixed relative to the center of mass. This four dimensional metric, found by one of us (K.L) and Lu, has the same asymptotic form as Eq. (37). We therefore conjecture that

the full metric coincides with the relative moduli space of two U(1) calorons on non-commutative $R^3 \times S^1$.

There are several pieces of evidence for this conjecture. Firstly, notice that both metrics become flat in the zero deformation parameter limit, which at least shows that our conjecture is consistent. Secondly, as shown in Ref. [12], the monopole moduli space of $SO(5)$ theory originates from that of two massive and two massless monopoles in $SU(4)$ gauge theory, whose symmetry is broken to $SU(2) \times U(1) \times SU(2)$. The identification of two massless monopoles in the $SU(4)$ case leads to the $SO(5)$ case. Thus one can get a two parameter family of four dimensional Hyperkähler spaces by independently holding fixed the two massless monopoles of the $SU(4)$ theory [22]. For example, when one of the parameters is sent to infinity, one naturally gets Dancer's metric [24], which describes two massive and one massless monopoles in $SU(3) \to SU(2) \times U(1)$. When the remaining parameter is sent to zero, this reduces to the double cover of the Atiyah-Hitchin. In contrast, when the two parameters coincide and are nonzero, one gets the four dimensional Lee-Lu metric that is of interest here.

The above discussion has a parallel in terms of three dimensional gauge theories. As Seiberg and Witten [30] have observed, the $N = 4$ supersymmetric three dimensional gauge theory with $SU(2)$ gauge group and one massive fundamental hypermultiplet has a quantum corrected Coulomb branch given by the Dancer metric. The case of interest here however is that of two, identified, fundamental hypermultiplets or, equivalently, a single hypermultiplet in the symmetric representation. For $SU(2)$, this is equivalent to a single massive adjoint hypermultiplet. As will be discussed in the following section, the Coulomb branch of this theory indeed coincides with the moduli space of two $U(1)$ non-commutative calorons. Related studies ALF spaces with the blow up of ADE singularities with the asymptotic space $R^3 \times S^1/\Gamma$, where Γ is the subgroup of $SU(2)$ can be found in [25].

One of the interesting questions is how many threshold bound states two U(1) calorons have. In the R^4 limit, there exists one as discussed in the previous section. It remains to be understood whether there are additional bound states which disappear in R^4 limit.

5 Calorons, Instantons and Three Dimensional Gauge Theories

We now turn to the construction of the moduli spaces of calorons and instantons as the quantum corrected Coulomb branch of three dimensional gauge theories. We will derive the gauge theories whose Coulomb branches yield the moduli space of calorons in A,D and E gauge groups. We will further show that in the case of A gauge groups only, there exists a mass parameter which deforms the Coulomb branch into the moduli space of non-commutative calorons. The limit in which the caloron moduli space becomes the instanton moduli space is shown to be the strong coupling limit of the gauge theory.

Let us start by recalling a few facts about instantons on ordinary (commutative) $R^3 \times S^1$. Specifically, we restrict attention to calorons in a simply-laced gauge group H which is broken to the maximal torus by a Wilson line around S^1. It was shown in [10, 26] that the resulting configurations are constructed of fundamental magnetic monopoles associated to the nodes of the *extended* Dynkin diagram of H. To each of the nodes of the Dynkin diagram, there is an associated root $\vec{\beta}_i$, $i = 0, \cdots, r = \mathrm{rank}\,(H)$. For $i = 1, \cdots, r$, these are the simple roots of H, while the lowest root of H, $\vec{\beta}_0$, is associated with the extended node. The topological charges of the caloron, which include both magnetic charge and Pontyagrin number, are determined by the choice of the number of monopoles of each type: these numbers are denoted by $n_i > 0$, $i = 0, \cdots, r$. In particular, the configurations which become the k-instanton solution in the decompactified R^4 limit are given by $n_i = k d_i$, where d_i are the Dynkin indices of H [26][3].

The three dimensional gauge theory which will reproduce the moduli space of calorons has $\mathcal{N} = 4$ supersymmetry (8 supercharges). The gauge group $\mathcal{G}$ and and the hypermultiplet matter content is determined by a quiver diagram based on the extended Dynkin diagram of H,

$$\mathcal{G} = \prod_{i=0}^{r} U(n_i) \,. \tag{39}$$

The gauge coupling constant of the $U(n_i)$ factor is e_i. Whenever there exists a line joining the ith and j^{th} node (i.e. whenever $\vec{\beta}_i \cdot \vec{\beta}_j \neq 0$ for $i \neq j$), the three-dimensional gauge theory also includes a hypermultiplet transforming in the $(\mathbf{n}_i, \bar{\mathbf{n}}_j)$ bi-fundamental representation of $U(n_i) \times U(n_j) \subset \mathcal{G}$.

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[3]Recall that for A_r, $d_i = 1$ for all i, for D_r, $d_i = 1$ for $i = 0, 1, r - 1, r$ and $d_i = 2$ for $i = 2, \cdots, r - 2$. Finally, for E_6, $d_i = 1, 1, 1, 2, 2, 2, 3$, for E_7, $d_i = 1, 1, 2, 2, 2, 3, 3, 4$ and for E_8, $d_i = 1, 2, 2, 3, 3, 4, 4, 5, 6$.

For the case in which $n_i = d_i$, where d_i are the Dynkin indices of H, the gauge theory above coincides with the Kronheimer gauge group which has been discussed in the context of mirror symmetry in three dimensional gauge theories in [27]. For the case in which $n_i = kd_i$ for $k > 1$, the gauge theory is very similar to those discussed in [28]. We will comment on this at the end of this section. Further discussion on the relationship between caloron moduli spaces and the vacuum moduli spaces of gauge theories can be found in [29, 13].

We are interested in computing the low-energy dynamics on the Coulomb branch of the theory. In such vacua the vacuum expectation values (VEVs) of the hypermultiplet scalars are set to zero, while the triplet of scalars, $\boldsymbol{\phi}$, that live in the vector multiplet acquire a VEV which, generically, breaks the gauge group $\mathcal{G}$ to its maximal torus. We assume such symmetry breaking does indeed occur. The massless fields on the Coulomb branch thus consist of $3N$ real scalars, ϕ^a, $a = 1, \cdots, N$ and N photons, where $N = \sum_{i=0}^{r} n_i$. Dualising the photons into N periodic scalars, σ^a, which we take to have period 2π, we are left with a sigma-model with a $4N$ dimensional target space. Supersymmetry requires the metric on this space to be hyperKähler. Although classically flat, this metric receives one-loop corrections, as well as non-perturbative corrections due to various instanton effects. While the latter are generally difficult to calculate, the former result in a toric hyperKähler metric given by,

$$ds^2 = g_{ab}d\boldsymbol{\phi}^a \cdot d\boldsymbol{\phi}^b + (g^{-1})^{ab}\left(d\sigma_a + \mathbf{W}_{ac} \cdot d\boldsymbol{\phi}^c\right)\left(d\sigma_b + \mathbf{W}_{bd} \cdot d\boldsymbol{\phi}^d\right) \tag{40}$$

with

$$g_{aa} = \frac{1}{\tilde{e}_a^2} - \sum_{b \neq a} \frac{\vec{\alpha}_a \cdot \vec{\alpha}_b}{|\boldsymbol{\phi}^a - \boldsymbol{\phi}^b|} \,,$$
$$g_{ab} = \frac{\vec{\alpha}_a \cdot \vec{\alpha}_b}{|\boldsymbol{\phi}^a - \boldsymbol{\phi}^b|} \quad (a \neq b) \,,$$

where $\vec{\alpha}_a = \vec{\beta}_i$ and $\tilde{e}_a = e_i$ if the corresponding massless fields ϕ^a and σ^a arise from the $U(n_i)$ factor of the gauge group. Finally, $\boldsymbol{\nabla} \times \mathbf{W}_{ab} = \boldsymbol{\nabla} g_{ab}$. When $n_i = 1$ for all i, the metric is complete and exact. For $n_i \geq 2$, the metric has singularities which are resolved by instanton corrections. In this case, the resulting metric is no longer of the toric form (40) and, in particular, the tri-holomorphic isometries arising from constant shifts in σ^a are broken by the instanton corrections.

For the special case $n_0 = 0$, the gauge group $\mathcal{G}$ is based on the non-affine Dynkin diagram and the above metric coincides with the asymptotic metric on the moduli space of monopoles of gauge

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group H, with magnetic charge $\vec{g} = \sum_{i=1}^{r} n_i \vec{\beta}_i$ [30]. The generalisation of this result is that for $n_0 \neq 0$, Eq. (40) agrees with the asymptotic metric on the moduli space of calorons. For gauge theories built on A_r Dynkin diagrams, this fact was pointed out some time ago [10]. The gauge coupling constants are related to the masses of the monopoles which, in turn, are determined by the VEV of the Wilson line, together with the radius, R, of the S^1. In particular, we have

$$\frac{1}{2\pi R} = \sum_{i=0}^{r} \frac{d_i}{e_i^2} \ . \tag{41}$$

(Here the dimension of both sides is matched by a reference scale needed for comparing two scales of the equation.) One may ask whether there is any natural deformation of the Coulomb branch metric (40) within the context of the three dimensional gauge theory. Such deformations arise through mass terms for the hypermultiplets and, as we shall see, lead to moduli spaces of calorons on non-commutative spaces. This observation has been previously made in [13]. It is simple to see that for the D and E Dynkin diagrams, all such mass terms may be absorbed through shifts of the vector multiplet scalars ϕ. This appears to tie in with the fact that there is no known non-commutative version of $SO(2r)$ and E_r gauge theories. Similarly, for A_r Dynkin diagrams, no such mass term is allowed if, say, $n_0 = 0$. Again, this dove-tails nicely with the fact that the monopole moduli space remains unchanged in the non-commutative theory. However, if $n_i \neq 0$ for all $i = 0, \cdots, r$, then we may use the freedom to shift ϕ to ensure that all but one of the bi-fundamental hypermultiplets has zero mass. This final hypermultiplet which, for $r \geq 2$, we may choose to be the one transforming in the $(\mathbf{n}_0, \bar{\mathbf{n}}_1)$ representation, has a triplet of mass parameters $\mathbf{m}$. The Coulomb branch metric for $r \geq 2$ is then given by (40), with

$$g_{aa} = \frac{1}{\tilde{e}_a^2} - \sum_{b \neq a} \frac{\vec{\alpha}_a \cdot \vec{\alpha}_b}{|\phi^a - \phi^b + \mathbf{m}_{ab}|} \ ,$$

$$g_{ab} = \frac{\vec{\alpha}_a \cdot \vec{\alpha}_b}{|\phi^a - \phi^b + \mathbf{m}_{ab}|} \quad (a \neq b) \ ,$$

with

$$\mathbf{m}_{ab} = +\mathbf{m} \quad \text{if } \vec{\alpha}_a = \vec{\beta}_0 \text{ and } \vec{\alpha}_b = \vec{\beta}_1 \ ,$$

$$\mathbf{m}_{ab} = -\mathbf{m} \quad \text{if } \vec{\alpha}_a = \vec{\beta}_1 \text{ and } \vec{\alpha}_b = \vec{\beta}_0 \ ,$$

$$\mathbf{m}_{ab} = 0 \quad \text{otherwise} \ .$$

This is the asymptotic metric on the moduli space of non-commutative calorons. Note that for the case of $r = 1$ ($SU(2)$), the metric does not take the above form: there are now two hypermultiplets

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in the $(\mathbf{n}_0, \bar{\mathbf{n}}_1)$ representation, of which one is massless and one is of mass $\mathbf{m}$. The corresponding metric is given by,

$$g_{aa} = \frac{1}{\tilde{e}_a^2} - \sum_{b \neq a} \frac{1}{2} \vec{\alpha}_a \cdot \vec{\alpha}_b \left(\frac{1}{|\phi^a - \phi^b + \mathbf{m}_{ab}|} + \frac{1}{|\phi^a - \phi^b|} \right),$$

$$g_{ab} = \frac{1}{2} \vec{\alpha}_a \cdot \vec{\alpha}_b \left(\frac{1}{|\phi^a - \phi^b + \mathbf{m}_{ab}|} + \frac{1}{|\phi^a - \phi^b|} \right) \quad (a \neq b)$$

with $\mathbf{m}_{ab}$ given as above.

We now turn to the main focus of this paper: non-commutative calorons in the $U(1)$ gauge theory. Note that, while the above construction was for calorons in gauge group $U(r+1)$ for $r \geq 1$, it may be extended to the $r = 0$ case. The gauge theory describing N $U(1)$ calorons is a $U(N)$ gauge group with a single adjoint hypermultiplet of mass $\mathbf{m}$. The perturbative metric is this time given by,

$$g_{aa} = \frac{1}{\tilde{e}^2} - \sum_{b > a} \left(\frac{2}{|\phi^a - \phi^b|} - \frac{1}{|\phi^a - \phi^b + \mathbf{m}|} - \frac{1}{|\phi^a - \phi^b - \mathbf{m}|} \right),$$

$$g_{ab} = \left(\frac{1}{|\phi^a - \phi^b|} - \frac{1}{|\phi^a - \phi^b + \mathbf{m}|} \right) \quad (a \neq b) . \tag{42}$$

The metric has a single $U(1)$ isometry which rotates ϕ fixing $\mathbf{m}$. The isometry is holomorphic with respect to a preferred complex structure. Note that the tri-holomorphic $U(1)$ isometry of the perturbative metric that results from shifts of σ is broken by instanton effects. For $N = 2$, one may factor out the center-of-mass to find the metric (37) with the appropriate normalization of the coefficients and the identification $\mathbf{m} \sim (0, 0, \zeta/2R)$.

It is interesting to examine various limits of the above metric. For instance, as $\mathbf{m} \to \infty$, with e^2 held fixed, the $U(1)$ holomorphic isometry is enhanced to an $SU(2)$ isometry which rotates the three complex structures. In this limit, the metric becomes the moduli space of N monopoles in $SU(2)$ gauge group which, for $N = 2$ is the Atiyah-Hitchin manifold. This is the limit where two plus heads of two dipoles are close to each other and separated far from two negative heads of two dipoles, and so the moduli space becomes that of two indentical monopoles. In general, in the limit of large non-commutivity, the moduli space of N non-commutative $U(k)$ calorons can take several different form. In the above metric (42) in the infinite $\mathbf{m}$ limit becomes the moduli space of $(N, N, \cdots, N)$ $SU(k+1)$ monopoles. This is the limit where dipoles are arranged so that one

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group negative heads of $U(1)$ charge is close to another group of positive heads of the same charge. One could take another limit where all positive heads are together. In this case the positive heads of different $U(1)$ would not interact and so one ends of k copies of the N moduli space of $SU(2)$ monopoles.

A less well understood limit is $e^2 \to \infty$. In this regime, the non-perturbative instanton corrections are no longer sub-leading and we have little control over the metric. Nevertheless, using equation (41), we see that this is the decompactification limit in which the Coulomb branch becomes the moduli space of N $U(1)$ instantons. From the results of section 3, we therefore find the result that the Coulomb branch of $SU(2)$ with a single massive adjoint hypermultiplet becomes Eguchi-Hanson in the strong coupling limit. It would be interesting re-derive this result using purely field theory methods. In particular, in this limit the metric develops an enhanced $SU(2)$ tri-holomorphic isometry. Such symmetry enhancement, while not at all uncommon [27], remains poorly understood from field theoretic considerations. It may be worth noting however that usually the symmetry group is enhanced from abelian factors to a non-abelian group of the same rank. In the present case, there is no tri-holomorphic isometry of the Coulomb branch metric for finite coupling constant.

Finally, let us discuss the caloron decompactification limit, $R \to \infty$, for general simply-laced gauge group H. As explained in [26], this leads to a finite action instanton solution of charge k precisely if the magnetic charges are chosen such that $n_i = kd_i$. Using (41), the decompactification limit of the caloron moduli space coincides with the strong coupling limit of the three dimensional gauge theory, $e_i \to \infty$. In fact, the strong coupling limit of gauge theories very similar to these have been arisen previously in the study of mirror symmetry [27, 28]. In particular, the authors of [27] discuss the $k = 1$ case. However, we find a subtle disagreement with the $k \geq 2$ generalisation discussed in [28], the resolution of which leads to a intriguing new prediction for the strong coupling dynamics of the gauge theory. Let us firstly restrict attention once more to the A_r gauge groups. We will make a few comments on the D and E cases at the end. The general mirror pairs discussed in [28] are

Theory A: $U(k)$ gauge group with a N fundamental hypermultiplets and a single adjoint hypermultiplet. There is a Fayet-Iliopoulos parameter (FI) ζ, an adjoint mass parameter M and fundamental mass parameters m_i, $i = 1, \cdots, N$, where we may choose $\sum_{i=1}^{N} m_i = 0$. The gauge

coupling constant is e.

Theory B: $\prod_{i=1}^{N} U(k_i)$ gauge group where $k_i = k$ for all $i = 1, \cdots, N$. There is a bi-fundamental hypermultiplet in each $(\bar{k}_i, k_{i+1})$ representation for $i = 1, \cdots, N$ [4]. The gauge group $U(k_1)$ has a further fundamental hypermultiplet transforming in k_1. There is a single mass parameter $\tilde{m}$ for this additional hypermultiplet transforming in the $(\bar{k}_N, k_1)$ representation, while all remaining hypermultiplets have no mass parameters. There are also N FI parameters, $\tilde{\zeta}_i$. The gauge coupling constants are $\tilde{e}_i$.

Note that Theory B is very similar to the theory that we introduced whose Coulomb branch coincides with the caloron moduli space if one chooses $n_i = k_i = k$. The two theories differ however by the presence of the fundamental hypermultiplet in Theory B. The Higgs branches of Theory A and Theory B, which arise as hyperkahler quotients, are both known in the mathematics literature. They exist fully only when $M = m_i = \tilde{m} = 0$, so let us concentrate on the theories with this restriction. Firstly, as discussed in section 2, the Higgs branch of Theory A is the moduli space of k instantons in a $U(N)$ gauge theory, $\mathcal{M}_{k,U(N)}$. Turning on the FI parameter ζ deforms this space to the moduli space of instantons in a non-commutative gauge theory.

In contrast, the Higgs branch of Theory B is the Hilbert scheme of k points in the A_{N-1} ALE space. We denote this by $Hilb^k(A_{N-1})$. This space is the smooth resolution of the symmetric product $S^k(A_{N-1}) = A_{N-1}^k/S_k$. The differences in the FI parameters, $\tilde{\zeta}_i - \tilde{\zeta}_{i+1}$, determine the size of the two-spheres which sit inside the blown-up A_{N-1} ALE. The sum of the FI parameters, $\sum_i \tilde{\zeta}_i$ determines the blow-up of the quotient singularities.

To summarise,

$$\text{Higgs}_A = \mathcal{M}_{k,U(N)}, \qquad \text{Higgs}_B = Hilb^k(A_{N-1}). \tag{43}$$

The statement of mirror symmetry in these theories is that in the infra-red limit $e \to \infty$ and $\tilde{e}_i \to \infty$, the Coulomb branches are also given by these spaces,

$$\text{Coulomb}_A = Hilb^k(A_{N-1}), \tag{44}$$

$$\text{Coulomb}_B = \mathcal{M}_{k,U(N)}, \tag{45}$$

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[4]we set $k_{N+1} = k_1$

where the mirror map between parameters is given by,

$$M = \sum_{i=1}^{N} \tilde{\zeta}_i \, , \quad \zeta = \tilde{m} \, , \quad m_i = \tilde{\zeta}_i - M/N \, . \tag{46}$$

Note in particular for $N = 1$, the theory is self-mirror and the above statement reduces to $\mathcal{M}_{k,U(1)} = Hilb^k(\mathbf{R}^4)$

While much evidence was presented in [28] for the first of these claims (44), much less is known about the latter (45). In particular, explicit comparisons of the metrics are hard to perform as instanton corrections are no longer sub-leading in the strong coupling limit. Nevertheless, if we accept the mirror symmetry proposal of [28], we are left with a situation in which both Theory B and the caloron theory introduced at the beginning of this chapter both yield the instanton moduli space in the strong coupling limit. While it is obvious that at finite e_i, the two Coulomb branches differ, it appears that the presence of the additional fundamental hypermultiplet of Theory B plays no role in the strong coupling dynamics. While such statements are known to be true for abelian theories (see for example [31]), the correspondence here requires remarkable cancellations between the instanton expansion of the two theories.

In fact, evidence for this correspondence arises from the brane construction of these models [32]. One can consider Theory A as the world-volume dynamics of k D3-branes in directions 0126 with the x^6 direction compactified. There is a single NS5-brane in the 012345 direction and N D5-branes in the 012789 directions. As usual, performing an S-duality results in Theory B. However, one could just as easily construct Theory A without the NS5-brane. S-duality then yields the caloron theory introduced at the beginning of this section. While the field content is unchanged, the removal of the NS5-brane does remove certain parameters. In Theory A, the adjoint mass parameter is not available once the NS5-brane is removed. From the mirror map, the sum of FI paramters of Theory B is unavailable in the dual picture.

While the above discussion was for the A_r group of theories, a similar discussion holds for the D_r group. For the E_r series, the mirror theory (i.e the equivalent of Theory A) has no known Lagrangian description and the point is moot.

Finally we comment that in abelian theories, the possibility of adding a hypermultiplet without

changing the strong coupling dynamics has been used in [31] to give an elegant formalism of mirror symmetry in these theories. It would be intriguing if the observation here would yield a similar construction for non-abelian theories.

Acknowledgements

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